NPRE 447 & 521
Interaction of Radiation with Matter
Homework Assignments

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1 Homework 1

Attention:

1. 447 students should ONLY solve the problems without asterisks, and will not receive extra credit for solving the problems with asterisks.

   521 students should solve ALL problems.

2. Submit your homework pdf file to gradescope.com. The Entry Code is V8KYVG.

3. Explanation of the score: our brains typically consume about 0.2 Calories per minute. When actively thinking, our brains can kick it up to burning about 1 Calorie per minute. So instead of assigning each question with points, I will assign with Calories. For example, if a problem is given 10 Calories, it means you will need to burn about 10 Calories to solve the problem and the estimated time to solve the problem is about 10 minutes.

4. Because of the breath and depth of the content of the course, it is only possible to cover the essence during the lectures. One must read the relevant chapters in the textbooks to learn the details and gain deeper understandings. Some homework problems should be considered as supplementary or self-learning materials.

Readings:

D. J. Griffiths and D. F. Schroeter, *Introduction to Quantum Mechanics*, 3rd edition, Cambridge University Press (2018).

- The entire Chapter 1
- Section 2.1 and Section 2.2
- * Section 2.6

1.1

Griffiths and Schroeter: Page 16, Problem 1.5. (30 Calories)

1.2

Griffiths and Schroeter: Page 76, Problem 2.37. (30 Calories)

1.3 *

Study Section 2.6 "The Finite Square Well" in Griffiths and Schroeter. Then consider a particle with energy E in rectangular barrier potential:

$$V(x) = \begin{cases} V_0, & |x| \leq a \\ 0, & |x| > a \end{cases}$$

where $a > 0$ and $V_0 > 0$. For each case $0 < E < V_0$, $E = V_0$, and $E > V_0$, solve and sketch the wave functions, calculate the transmission and reflection coefficients T and R, and find out the behavior of T and R when $V_0 \to \infty$. (60 Calories)

2 Homework 2

Readings:

D. J. Griffiths and D. F. Schroeter, *Introduction to Quantum Mechanics*, 3rd edition, Cambridge University Press (2018).

- Section 2.3.2

2.1

The Hamiltonian of a 1-dimensional quantum harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

1. Solve the time-independent Schrödinger equation using the analytic method to show the eigen state wave functions $\psi_n(x)$ and the corresponding eigen energy E_n are

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{\lambda}{\pi}\right)^{1/4} \exp\left(-\frac{1}{2}\lambda x^2\right) H_n\left(\sqrt{\lambda}x\right)$$

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

   where $\lambda = m\omega/\hbar$, $H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$ are the Hermite polynomials. (30 Calories)

2. For $n = 0, 1, 2$, write down $\psi_n(x)$ explicitly, produce computer plots of the wave functions $\psi_n(x)$ and the square of the wave functions $|\psi_n(x)|^2$ on top of the potential. Attach your code. (15 Calories)

3. For $n = 0, 1, 2$, check the orthogonality of $\psi_n(x)$ by explicit integrations. (15 Calories)

4. For $n = 0, 1, 2$, compute $\langle x\rangle$, $\langle x^2\rangle$, σ_x, $\langle p\rangle$, $\langle p^2\rangle$, σ_p. Compute the uncertainty relation quantity $\sigma_x\sigma_p$. Check whether the uncertainty relation is satisfied. Which state is closest to the uncertainty limit? (15 Calories)

5. For $n = 0, 1, 2$, compute the expectation values of the kinetic energy $\langle T\rangle$ and the potential energy $\langle V\rangle$ by explicit integrations. (15 Calories)

6. If the particle starts out in the initial state

$$\Psi(x, 0) = A\left[3\psi_0(x) + 4\psi_1(x)\right]$$

   what is A? what is $\Psi(x,t)$? If we measure the energy of this particle, what values may we get and with what probabilities? What's the expectation value of $\langle H\rangle$? (20 Calories)

2.2

Griffiths and Schroeter: Page 54, Problem 2.14. (20 Calories)

3 Homework 3

Readings:

D. J. Griffiths and D. F. Schroeter, *Introduction to Quantum Mechanics*, 3rd edition, Cambridge University Press (2018).

- Section 2.3.1

3.1

Let's reconsider the 1-dimensional quantum harmonic oscillator

$$H = T + V = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

using the ladder operators

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right)$$
$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$$

1. Solve x and p in terms of a and $a^\dagger$. Show that the Hamiltonian operator can be written in terms of the ladder operators $H = (N + \frac{1}{2})\hbar\omega$, where $N = a^\dagger a$. (10 Calories)
2. Compute $[a, a^\dagger]$, $[N, a]$, and $[N, a^\dagger]$, where $N = a^\dagger a$. (15 Calories)
3. Show that the energy eigenstates are $|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$ and the energy eigenvalues are $E_n = (n + \frac{1}{2})\hbar\omega$. (20 Calories)
4. Sketch the wave functions of the first three eigen states (n = 0, 1, and 2) in a harmonic potential. (5 Calories)
5. For the n^{th} energy eigenstate $|n\rangle$, compute the expectation values of $\langle n|x|n\rangle$, $\langle n|x^2|n\rangle$, $\langle n|p|n\rangle$, $\langle n|p^2|n\rangle$, and the uncertainty relation quantity $\sigma_x\sigma_p$. (10 Calories)
6. For the n^{th} energy eigenstate $|n\rangle$, compute the expectation values of the kinetic energy $\langle n|T|n\rangle$ and the potential energy $\langle n|V|n\rangle$. (10 Calories)
7. If the particle starts out in the initial state

$$|\Psi(t = 0)\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

   what is its state at time t? If we perform measurements of the total energy of the system at time t, what do we get? Do they depend on time? (10 Calories)
8. Give one physical example of such quantum harmonic oscillators. Hint: one of such examples is a nuclear material. (10 Calories)

3.2

Griffiths and Schroeter: Page 60, Problem 2.17. (10 Calories)

4 Homework 4

Readings:

D. J. Griffiths and D. F. Schroeter, *Introduction to Quantum Mechanics*, 3rd edition, Cambridge University Press (2018).

- Section 2.4
- Section 2.5

4.1

Griffiths and Schroeter: Page 69, Problem 2.22. (30 Calories)

4.2

In atomic and nuclear physics, many systems can be described by pseudo-potentials, such as the Fermi pseudo-potential $V(r) = \frac{2\pi\hbar^2}{m} b\delta(r)$ used in describing neutron scattering processes, where b is the bound scattering length. Let's consider a particle with mass m in the 1-dimensional δ-potential

$$V(x) = -\alpha\delta(x)$$

where $\alpha > 0$ is the strength of the potential with the dimension of [EL].

1. Compute the bound state ($E < 0$) wave function and the corresponding eigen energy level(s). (30 Calories)
2. Compute the probability current of the wave function in Part 1. Explain the physical meaning of the result. (5 Calories)
3. Compute the transmitted and reflected wave functions of an incoming plane wave $\psi_i(x) = Ae^{ikx}$ ($E > 0$, scattering state). (30 Calories)
4. Compute the probability current on both sides of the potential in Part 3. Explain the physical meaning of the result. (10 Calories)
5. Compute the transmission coefficient T and reflection coefficient R in Part 3. (10 Calories)
6. Perform dimensional analysis on the eigen energy computed in Part 1, and the transmission and reflection coefficients computed in Part 5. (10 Calories)
7. What's the asymptotic behavior of T and R if the potential is very deep, i.e. $\alpha \to \infty$? Explain the physical meaning. (5 Calories)
8. If we flip the potential, i.e. let $\alpha < 0$, do we still have bound states? How about the scattering state? (10 Calories)

4.3

Griffiths and Schroeter: Page 60, Problem 2.20. (40 Calories)

4.4

1. From the continuity equation
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$
derive the probability current
$$\mathbf{J} = \frac{i\hbar}{2m}(\Psi \nabla \Psi^* - \Psi^* \nabla \Psi) = \frac{\hbar}{m} \mathrm{Im}\{\Psi^* \nabla \Psi\}$$
(10 Calories)

2. Compute the probability current for the plain wave $\Psi(x, t) = e^{i(kx - \omega t)}$. (10 Calories)

3. Compute the probability current for the spherical wave $\Psi(\mathbf{r}, t) = e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}/r$. (10 Calories)

4.5 *

Griffiths and Schroeter: Page 22, Problem 1.14. (30 Calories)

5 Homework 5

Readings:

D. J. Griffiths and D. F. Schroeter, *Introduction to Quantum Mechanics*, 3rd edition, Cambridge University Press (2018).

- Section 3.1
- Section 3.2
- Section 3.3
- Section 3.4
- Section 3.6

5.1

Griffiths and Schroeter: Page 120, Problem 3.25. (30 Calories)

5.2

Griffiths and Schroeter: Page 120, Problem 3.26. (30 Calories)

5.3

Griffiths and Schroeter: Page 128, Problem 3.44. (30 Calories)

5.4 *

Consider a two-state system (e.g. a neutron or a proton with spin 1/2) with the following Hamiltonian

$$H = \epsilon_1(\left|\uparrow\right\rangle\left\langle\uparrow\right| + \left|\downarrow\right\rangle\left\langle\downarrow\right|) + \epsilon_2(\left|\uparrow\right\rangle\left\langle\downarrow\right| + \left|\downarrow\right\rangle\left\langle\uparrow\right|)$$

1. What is the matrix representation of H in the $\left|\uparrow\right\rangle$ and $\left|\downarrow\right\rangle$ basis? (10 Calories)
2. Are $\left|\uparrow\right\rangle$ and $\left|\downarrow\right\rangle$ the eigen states of this system? If not, what are the eigen states and their corresponding eigen energies? (10 Calories)
3. If the system starts out in state $\left|\Psi(0)\right\rangle = A(3\left|\uparrow\right\rangle + 4\left|\downarrow\right\rangle)$ for some constant A,
   (a) Determine the constant A? (10 Calories)
   (b) What is the time-dependent solution $\left|\Psi(t)\right\rangle$? (10 Calories)
   (c) If we perform a measurement of the energy of the particle, what values may we get and with what probabilities? What's the expectation value $\langle H\rangle$? (10 Calories)

6 Homework 6

Readings:

D. J. Griffiths and D. F. Schroeter, *Introduction to Quantum Mechanics*, 3rd edition, Cambridge University Press (2018).

- Section 3.5

6.1

Griffiths and Schroeter: Page 124, Problem 3.33. (30 Calories)

6.2

Griffiths and Schroeter: Page 108, Problem 3.14. (30 Calories)

6.3

Macroscopic Quantum World: The Planck constant (reduced) $\hbar \approx 1\times 10^{-34}\ J\cdot s$ plays a fundamental role in quantum mechanics. Imagine that one day you are teletransported to another universe, where the reduced Planck constant is 10^{34} times larger, i.e. $\hbar \approx 1\ J\cdot s$. Let's picture what strange phenomena you would expect.

1. First, derive the uncertainty principle

$$\sigma_A \sigma_B \geq \frac{1}{2} \left| \langle [A, B] \rangle \right|$$

   and state its the significance. (20 Calories)

2. Suppose you have a jar of candies in this quantum world. When you open the jar, estimate the escape velocities of the candies. Do you need to be careful when opening the jar? (A typical weight of a candy is 1 gram. A typical size of a jar is 10 cm.) (10 Calories)

3. Use your imagination, describe another two strange phenomena you would expect in this quantum world. Be as quantitative as possible. (20 Calories)

6.4 *

Griffiths and Schroeter: Page 124, Problem 3.34. (30 Calories)

7 Homework 7

Readings:

D. J. Griffiths and D. F. Schroeter, *Introduction to Quantum Mechanics*, 3rd edition, Cambridge University Press (2018).

- Section 4.1
- Section 4.2
- Section 4.3

7.1

Read Chapter 4.1, 4.2, and 4.3 of Griffiths and Schroeter carefully and study how to solve the eigenstates of hydrogen atom. Summarize the solutions of the eigen wavefunctions, energy, and angular momentum. Explain their physical meanings. (No need to solve the equations.) (30 Calories)

7.2

1. Use the recursive relation to work out the radial wave functions of the Hydrogen atom $R_{10}(r)$, $R_{20}(r)$, and $R_{21}(r)$. Normalize them. (30 Calories)
2. For each of the above states, compute $\langle r \rangle$ and $\langle r^2 \rangle$. Express the the answers in terms of the Bohr radius. (30 Calories)

7.3 Spherical harmonics

Produce pseudocolor plots (also called density plots) of the real, imaginary, and absolute values of the spherical harmonics $Y_l^m(\theta, \phi)$ on a unit sphere for $l = 0, 1, 2, 3$ and all the allowed m. The color should be mapped to the values of $Y_l^m(\theta, \phi)$. Attach your code. (20 Calories)
Example: https://en.wikipedia.org/wiki/Spherical_harmonics#/media/File:Spherical_harmonics.png
Tips:

- In Mathematica, try `SphericalPlot3D[]` by setting $r = 1$ and with the `ColorFunction` option.
- In python/matplotlib, try `plot_surface()` by setting $r = 1$ during the coordinate conversion (`sph2cart()`) and with the the `surface color` option.
- In Matlab, try `surf()` by by setting $r = 1$ during the coordinate conversion (`sph2cart()`) and with the `color` option.

7.4 Hydrogen atomic orbital

1. *Optional:* Produce 3D translucent pseudocolor plots (also called density plots) of the real, imaginary, absolute value, and absolute value squared (probability density) of the hydrogen atomic orbitals $\psi_{nlm}(r,\theta,\phi)$ for $n = 0, 1, 2, 3$ and all the allowed l and m. The color should be mapped to the values of the function. Attach your code. (0 Calories)

   Example: https://en.wikipedia.org/wiki/File:Atomic-orbital-clouds_spdf_m0.png

   Tips:

   - In Mathematica, try `DensityPlot3D[]`.

2. Produce 2D cut (along the $z-x$ plane) pseudocolor plots (also called density plots) of the real, imaginary, absolute value, and absolute value squared (probability density) of the hydrogen atomic orbitals $\psi_{nlm}(r,\theta,\phi)$ for $n = 0, 1, 2, 3$ and all the allowed l and m. The color should be mapped to the values of the function. Attach your code. (20 Calories)

   Example: https://en.wikipedia.org/wiki/Atomic_orbital#/media/File:Hydrogen_Density_Plots.png

   Tips:

   - In Mathematica, try `DensityPlot[]`.
   - In python/matplotlib, try `pcolor()`.
   - In Matlab, try `pcolor()`.

3. Produce 3D isosurface plots of the real, imaginary, absolute value, and absolute value squared (probability density) of the hydrogen atomic orbitals $\psi_{nlm}(r,\theta,\phi)$ for $n = 0, 1, 2, 3$ and all the allowed l and m. Attach your code. (20 Calories)

   Example: https://en.wikipedia.org/wiki/Atomic_orbital#Orbitals_table

   Tips:

   - In Mathematica, try `ContourPlot3D[]`.
   - In python/mayavi, try `iso_surface()`.
   - In Matlab, try `isosurface()`.

7.5

Griffiths and Schroeter: Page 155, Problem 4.19. (20 Calories)

8 Homework 8

Readings:

D. J. Griffiths and D. F. Schroeter, *Introduction to Quantum Mechanics*, 3rd edition, Cambridge University Press (2018).

- Section 4.3

8.1

1. Prove $[L_x, L_y] = i\hbar L_z$ (5 Calories)
2. Prove $[L^2, L_z] = 0$ (5 Calories)
3. Use the ladder operators, prove $L^2|l,m\rangle = \hbar^2 l(l+1)|l,m\rangle$ and $L_z|l,m\rangle = \hbar m|l,m\rangle$ (30 Calories)
4. Prove $L_\pm|l,m\rangle = \hbar\sqrt{(l \mp m)(l \pm m+1)}|l,m \pm 1\rangle = \hbar\sqrt{l(l+1) - m(m \pm 1)}|l,m \pm 1\rangle$ (30 Calories)

8.2

Write down the angular momentum operator $\mathbf{L}$ in spherical coordinates. Show that

$$L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$$

and

$$L_z = -i\hbar \frac{\partial}{\partial\phi}$$

Therefore, the eigen equations of them are indeed the angular equations obtained from the separation of variables. (30 Calories)

9 Homework 9

Readings:

D. J. Griffiths and D. F. Schroeter, *Introduction to Quantum Mechanics*, 3rd edition, Cambridge University Press (2018).

- Section 4.4

9.1 *

Suppose a neutron (or an electron, or a proton) is in the spin state $A\begin{pmatrix} i \\ 1 \end{pmatrix}$.

1. What is the normalization factor A? (5 Calories)
2. If we measure S_x, what values do we get? What is the probability of each value? (5 Calories)
3. If we measure S_y, what values do we get? What is the probability of each value? (5 Calories)
4. If we measure S_z, what values do we get? What is the probability of each value? (5 Calories)
5. What are the expectation values of $\langle S_x \rangle$, $\langle S_x^2 \rangle$, $\langle S_y \rangle$, $\langle S_y^2 \rangle$, $\langle S_z \rangle$, and $\langle S_z^2 \rangle$? Verify the uncertainty relations. (20 Calories)

9.2 *

Consider a neutron in a uniform magnetic field along the z axis $\mathbf{B} = B_0 \hat{z}$.

1. If the initial state is $|\psi(0)\rangle = \begin{pmatrix} \cos\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} \end{pmatrix}$, where α is a known constant, what is the time evolution of the state $|\psi(t)\rangle$? (20 Calories)
2. What are the expectation values of $\langle S_x \rangle$, $\langle S_y \rangle$, and $\langle S_z \rangle$? Note that $\omega = \gamma B_0$ is the Larmor frequency, where γ is the gyromagnetic ratio. (20 Calories)

9.3 *

Construct the spin matrices S_x, S_y, and S_z for a particle with spin $s = 1$ (for example, a deuteron). (30 Calories)

9.4 *

Consider the total spin angular momentum of two spin 1/2 particles, such as a neutron and a proton.

1. What are the singlet and triplet states? (10 Calories)
2. Show that $|1, 1\rangle$ and $|1, 0\rangle$ are eigen states of S_z. (10 Calories)
3. Show that $|1, 1\rangle$ and $|1, 0\rangle$ are eigen states of S^2. (10 Calories)
4. Apply S_- to $|1, 1\rangle$ to confirm you get $\sqrt{2}\hbar|1, 0\rangle$. (10 Calories)
5. Apply S_- to $|1, 0\rangle$ to confirm you get $\sqrt{2}\hbar|1, -1\rangle$. (10 Calories)

10 Homework 10

Readings:

D. J. Griffiths and D. F. Schroeter, *Introduction to Quantum Mechanics*, 3rd edition, Cambridge University Press (2018).

- Section 10.1
- Section 10.2
- Section 10.3

10.1 *

What is *Quantum Entanglement*? Give one example of an entangled state. (10 Calories)

10.2 *

What is the *differential cross section* for scattering $\frac{d\sigma}{d\Omega}$? Explain its physical meaning. (20 Calories)

10.3 *

For thermal neutron scattering, the differential cross section turns out to be very simple. What is it? What approximations are invoked to derive the result? (20 Calories)

10.4 *

What does a positive scattering length mean? What does a negative scattering length mean? (20 Calories)