NPRE 446 Fall 2020 Homework 8 Solutions

8.1 Complex Scattering is inelastic scattering of a photon by a free, stationary electron.

8.2 Momentum lost by photon is gained by electron (conservation of momentum):

\[ \vec{h} \vec{k} - \vec{h} \vec{k}' = \vec{p}_e \]

Square both sides:

\[ h^2 k^2 + h^2 k'^2 - 2h^2 \vec{k} \vec{k}' \cos \theta = p_e^2 \]

This is the relativistic momentum of the electron after the collision. It is related to the total relativistic energy by:

\[ E^2 = p_e c^2 + m_e c^4 \]

So the total energy of the electron and photon after the scatter is:

\[ \sqrt{m_e c^2 + (h^2 k^2 + h^2 k'^2 - 2h^2 \vec{k} \vec{k}' \cos \theta)c^2} + \hbar \omega' \]

By conservation of energy, this is equal to the total relativistic energy of the photon + electron before the scatter:

\[ \hbar \omega + m_e c^2 \quad \text{assuming the electron starts at rest} \]

\[ \Rightarrow \hbar \omega + m_e c^2 = \hbar \omega' + \sqrt{m_e c^2 + (h^2 k^2 + h^2 k'^2 - 2h^2 \vec{k} \vec{k}' \cos \theta)c^2} \]

\[ \Rightarrow (\hbar \omega - \hbar \omega' + m_e c^2)^2 = m_e c^4 + (h^2 k^2 + h^2 k'^2 - 2h^2 \vec{k} \vec{k}' \cos \theta)c^2 \]

\[ \Rightarrow (h \vec{k} - h \vec{k}' + m_e c)^2 = m_e c^2 + h^2 k^2 + h^2 k'^2 - 2h^2 \vec{k} \vec{k}' \cos \theta \]
Dividing by $\hbar c^2$:
\[
\left(\frac{k - k' + \frac{mc}{\hbar}}{\hbar^2}\right)^2 = \frac{mc^2}{\hbar^2} + \frac{k^2 + k'^2 - 2kk' \cos \theta}{\hbar^2}
\]
\[
k^2 - k'k + \frac{2kmec}{\hbar} = \frac{2k'mec}{\hbar} + \frac{mc^2}{\hbar^2} - \frac{k^2 - k'^2}{\hbar^2} \cos \theta
\]
\[
\Rightarrow \frac{mc}{\hbar} (k - k') = kk' (1 - \cos \theta)
\]
\[
\Rightarrow \frac{1}{k} - \frac{1}{k'} = \frac{\hbar}{mc} (1 - \cos \theta)
\]
\[
\Rightarrow [\frac{1}{k'} - \frac{1}{k}] = \frac{2\pi \hbar}{mc} (1 - \cos \theta) = \gamma_c (1 - \cos \theta)
\]

8.1.3 Because the electron is initially at rest, its final kinetic energy is equal to the kinetic energy it picks up from the collision. Due to conservation of energy, this is simply the energy lost by the photon:

\[
T = \hbar \omega - (\frac{\hbar \omega}{2}) = \hbar \omega - 2 \pi mc
\]

We have an expression for $\gamma'$ from part 2:

\[
T = \hbar \omega - 2 \pi mc = \hbar \omega - \frac{2 \pi \hbar mc}{\gamma + 2 \gamma (1 - \cos \theta)} = \hbar \omega - \frac{2 \pi \hbar mc}{\gamma \omega + \frac{2 \pi \hbar mc}{\gamma} (1 - \cos \theta)}
\]

\[
\text{Multiply top + bottom by } \omega/c:\n\]
\[
\frac{\hbar^2 \omega^2 (1 - \cos \theta)}{mec^2} (1 - \frac{\hbar \omega}{mec^2} (1 - \cos \theta))
\]

\[
\frac{\hbar^2 \omega^2 (1 - \cos \theta)}{mec^2 (1 - \frac{\hbar \omega}{mec^2} (1 - \cos \theta))}
\]
\[ T = \frac{k \omega \alpha (1 - \cos \Theta)}{1 + \alpha (1 - \cos \Theta)} \text{ where } \alpha = \frac{k \omega}{M_0 c^2} \]

8.1.4 \( \alpha \) is the energy of the incoming photon, in units of electron rest mass energy (0.511 MeV). As \( \alpha \to 1 \), the scattering becomes elastic, but the photon loses a large amount of energy in the lab frame. In terms of \( \alpha \), the differential cross section is:

\[ \frac{d\sigma}{d\Omega} = \frac{r_0^2}{2} \left( \frac{1 + \cos \Theta}{1 + \alpha (1 - \cos \Theta)} \right)^2 \times \left[ 1 + \frac{\alpha^2 (1 - \cos \Theta)^2}{(1 + \cos \Theta) (1 + \alpha (1 - \cos \Theta))} \right] \]

Note this is if the photon is unpolarized.

At low energy, forward (0°) and backward (180°) non-collision (180°) are equally likely. At high energies, there is a forward peak.

8.1.5 The angular frequency of the scattered photon is related to its initial angular frequency and angle of scattering by:

\[ \omega' = \frac{\omega}{1 + \alpha (1 - \cos \Theta)} \]
when the maximum energy is given to the electron ($\Theta = \pi$), this simplifies to:

$$\omega' = \frac{1}{\omega + \frac{2h}{mc^2}}$$

The limit as incoming photon energy goes to infinity is:

$$\lim_{\omega \to \infty} E' = \lim_{\omega \to \infty} \frac{\hbar}{\omega} = \frac{\hbar}{\omega + \frac{2h}{mc^2}} = \frac{mc^2}{2} = 0.255 \text{ MeV}$$

Now, let's sketch the differential energy cross section:

![Graph showing differential energy cross section](image)

Note in the high $E'_y$ limit, the electron energy approaches $E_y - 0.255 \text{ MeV}$.

The "pileup" near $1.2 \text{ MeV}$ is due to the fact that cosine changes slowly in the region of $\Theta = \pi$. 
8.2.1 The photoelectric effect is when a photon is absorbed by an electron in an atom, resulting in the emission of the electron. It was found that the threshold was dependent on the wavelength, rather than the intensity of the incoming light, which is strong evidence of the particle nature of light.

8.2.2 An inner shell electron is more likely to be emitted because outer shell electrons appear as free electrons. A free electron cannot undergo the photoelectric effect because conservation of momentum + energy will not be satisfied.

8.2.3 The atom will de-excite by emission of an x-ray or an Auger electron.

8.3.1 Compton Scattering: \( \frac{1}{\hbar \omega} \) per electron

\[ \text{photoelectric effect: } \sigma \propto \frac{\gamma}{(\hbar \omega)^{2/3}} \]

\[ \text{pair production: } \sigma \propto Z^2 \ln \left( \frac{2\hbar \omega}{m_e c^2} \right) \text{ per atom} \]

8.3.2 A graph showing the relationship between the photoelectric effect and pair production with respect to photon energy (\( \hbar \omega \)).
8.33

[Diagram of a scenario with labels and arrows, including "Total" and "Compton recoil probability"]